

# SUBSPACE CONCENTRATION OF DUAL CURVATURE MEASURES OF SYMMETRIC CONVEX BODIES

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ABSTRACT. We prove a tight subspace concentration inequality for the dual curvature measures of a symmetric convex body.

## 1. INTRODUCTION

Let  $\mathcal{K}^n$  denote the set of convex bodies in  $\mathbb{R}^n$ , i.e., all convex and compact subsets  $K$  having a non-empty interior. The set of convex bodies having the origin as an interior point and the set of origin-symmetric convex bodies, i.e., those sets which satisfy  $K = -K$  are denoted by  $\mathcal{K}_o^n$  and  $\mathcal{K}_e^n$  respectively. For  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ , let  $\langle \mathbf{x}, \mathbf{y} \rangle$  denote the standard inner product and  $|\mathbf{x}| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$  the Euclidean norm. We write  $B_n$  for the  $n$ -dimensional Euclidean unit ball, i.e.,  $B_n = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq 1\}$  and  $S^{n-1}$  for its boundary. The  $k$ -dimensional Hausdorff-measure will be denoted by  $\mathcal{H}^k(\cdot)$  and instead of  $\mathcal{H}^n(\cdot)$  we will also write  $\text{vol}(\cdot)$  for the  $n$ -dimensional volume.

At the heart of the Brunn-Minkowski theory is the study of the volume functional with respect to the Minkowski addition of convex bodies. This leads to the theory of mixed volumes and, in particular, to the quermass-integrals  $W_i(K)$  of a convex body  $K \in \mathcal{K}^n$ . The latter may be defined via the classical Steiner formula, expressing the volume of the Minkowski sum of  $K$  and  $\lambda B_n$ , i.e., the volume of the parallel body of  $K$  at distance  $\lambda$  as a polynomial in  $\lambda$  (cf., e.g., [40, Sect. 4.2])

$$(1.1) \quad \text{vol}(K + \lambda B_n) = \sum_{i=0}^n \lambda^i \binom{n}{i} W_i(K).$$

A more direct geometric interpretation is given by Kubota's integral formula (cf., e.g., [40, Subsect. 5.3.2]), showing that they are – up to some constants – the means of the volumes of projections

$$(1.2) \quad W_{n-i}(K) = \frac{\text{vol}(B_n)}{\text{vol}_i(B_i)} \int_{G(n,i)} \text{vol}_i(K|L) \, dL, \quad i = 1, \dots, n,$$

where  $\text{vol}_i(\cdot)$  denotes the  $i$ -dimensional volume, integration is taken with respect to the rotation-invariant probability measure on the Grassmannian  $G(n, i)$  of all  $i$ -dimensional linear subspaces and  $K|L$  denotes the orthogonal projection onto  $L$ .

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A local version of the Steiner formula above leads to two important series of geometric measures, the surface area measures  $S_i(K, \cdot)$  and the curvature measures  $C_i(K, \cdot)$ ,  $i = 0, \dots, n-1$ , of a convex body  $K$ . Here we will only briefly describe the surface area measures since they may be considered as the “primal” counterpart to the dual curvature measures we are interested in.

To this end, we denote for  $\omega \subseteq S^{n-1}$  by  $\nu_K^{-1}(\omega) \subseteq \partial K$  the set of all boundary points of  $K$  having an outer unit normal in  $\omega$ . Moreover, for  $\mathbf{x} \in \mathbb{R}^n \setminus K$  let  $r_K(\mathbf{x}) \in \partial K$  be the point of  $K$  closest to  $\mathbf{x}$ . Then for a Borel set  $\omega \subseteq S^{n-1}$  and  $\lambda > 0$  we consider the local parallel body

$$(1.3) \quad B_K(\lambda, \omega) = \{\mathbf{x} \in \mathbb{R}^n : 0 < |\mathbf{x} - r_K(\mathbf{x})| \leq \lambda \text{ and } r_K(\mathbf{x}) \in \nu_K^{-1}(\omega)\}.$$

The local Steiner formula is now a polynomial in  $\lambda$  whose coefficients are (up to constants depending on  $i, n$ ) the surface area measures (cf., e.g., [40, Sect. 4.2])

$$(1.4) \quad \text{vol}(B_K(\lambda, \omega)) = \frac{1}{n} \sum_{i=1}^n \lambda^i \binom{n}{i} S_{n-i}(K, \omega).$$

They may also be regarded as the (right hand side) differentials of the quermassintegrals

$$(1.5) \quad \lim_{\epsilon \downarrow 0} \frac{W_{n-1-i}(K + \epsilon B_n) - W_{n-1-i}(K)}{\epsilon} = \int_{S^{n-1}} dS_i(K, \mathbf{u}).$$

Also observe that  $S_i(K, S^{n-1}) = n W_{n-i}(K)$ ,  $i = 0, \dots, n-1$ .

To characterize the surface area measures  $S_i(K, \cdot)$ ,  $i \in \{1, \dots, n-1\}$ , among the finite Borel measures on the sphere is a corner stone of the Brunn-Minkowski theory. Today this problem is known as the *Minkowski-Christoffel* problem, since for  $j = n-1$  and the surface area measure  $S_{n-1}(K, \cdot)$  it is the classical Minkowski problem and for  $j = 1$  it is the Christoffel problem. We refer to [40, Chapter 8] for more information and references.

There are two far-reaching extensions of the classical Brunn-Minkowski theory, both arising basically by replacing the classical Minkowski-addition by another additive operation (cf. [17]). The first one is the  $L_p$  addition introduced by Firey (see, e.g., [12]) which leads to the rich and emerging  *$L_p$ -Brunn-Minkowski theory* for which we refer to [40, Sect. 9.1, 9.2].

The second one, introduced by Lutwak [29], is based on the radial addition  $\widetilde{+}$  where  $\mathbf{x} \widetilde{+} \mathbf{y} = \mathbf{x} + \mathbf{y}$  if  $\mathbf{x}, \mathbf{y}$  are linearly dependent and  $\mathbf{0}$  otherwise. Considering the volume of radial additions leads to the *dual Brunn-Minkowski theory* (cf. [40, Sect. 9.3]) with dual mixed volumes, and, in particular, also with dual quermassintegrals  $\widetilde{W}_i(K)$  arising via a dual Steiner formula (cf. (1.1))

$$\text{vol}(K \widetilde{+} \lambda B_n) = \sum_{i=0}^n \lambda^i \binom{n}{i} \widetilde{W}_i(K).$$

In general the radial addition of two convex sets is not a convex set, but the radial addition of two star bodies is again a star body. This is one of the features of the dual Brunn-Minkowski theory which makes it so useful. The

celebrated solution of the Busemann-Petty problem is amongst the recent successes of the dual Brunn-Minkowski theory, cf. [13, 18, 44], and it also has connections and applications to integral geometry, Minkowski geometry, and the local theory of Banach spaces.

In analogy to Kubota's formula (1.2) the dual quermassintegrals  $\widetilde{W}_i(K)$  admit the following integral geometric representation as the means of the volumes of sections (cf. [40, Sect. 9.3])

$$\widetilde{W}_{n-i}(K) = \frac{\text{vol}(B_n)}{\text{vol}_i(B_i)} \int_{G(n,i)} \text{vol}_i(K \cap L) dL, \quad i = 1, \dots, n.$$

There are many more “dualities” between the classical and dual theory, but there were no dual geometric measures corresponding to the surface area or curvature measures. This missing link was recently established in the ground-breaking paper [25] by Huang, Lutwak, Yang and Zhang. Let  $\rho_K$  be the radial function (see Section 2 for the definition) of a convex body  $K \in \mathcal{K}_o^n$ . Analogous to (1.3) we consider for a Borel set  $\eta \subseteq S^{n-1}$  and  $\lambda > 0$  the set

$$\widetilde{A}_K(\lambda, \eta) = \{x \in \mathbb{R}^n : 0 \leq |x - \rho_K(x)x| \leq \lambda \text{ and } \rho_K(x)x \in \nu_K^{-1}(\eta)\}.$$

Then there also exists a local Steiner type formula of these local dual parallel sets [25, Theorem 3.1] (cf. (1.4))

$$\text{vol}(\widetilde{A}_K(\lambda, \eta)) = \sum_{i=0}^n \binom{n}{i} \lambda^i \widetilde{C}_{n-i}(K, \eta).$$

$\widetilde{C}_i(K, \eta)$  is called the *ith dual curvature measure* and they are the counterparts to the surface area measures  $S_i(K, \omega)$  in the dual Brunn-Minkowski theory. Observe that  $\widetilde{C}_i(K, S^{n-1}) = \widetilde{W}_{n-i}(K)$ . As the surface area measure (cf. (1.5)), the dual curvature measures may also be considered as differentials of the dual quermassintegrals, even in a stronger form (see [25, Section 4]). We want to point out that there are also dual surface area measures corresponding to the curvature measures in the classical theory (see [25]).

Huang, Lutwak, Yang and Zhang also gave an explicit integral representation of the dual curvature measures which allowed them to define more generally for  $q \in \mathbb{R}$  the  $q$ th dual curvature measure of a convex body  $K \in \mathcal{K}_o^n$  as [25, Def. 3.2]

$$(1.6) \quad \widetilde{C}_q(K, \eta) = \frac{1}{n} \int_{\alpha_K^*(\eta)} \rho_K(u)^q d\mathcal{H}^{n-1}(u).$$

Here  $\alpha_K^*(\eta)$  denotes the set of directions  $u \in S^{n-1}$ , such that the boundary point  $\rho_K(u)u$  belongs to  $\nu_K^{-1}(\eta)$ . The analog to the Minkowski-Christoffel problem in the dual Brunn-Minkowski theory is (cf. [25, Sect. 5])

*The dual Minkowski problem.* Given a finite Borel measure  $\mu$  on  $S^{n-1}$  and  $q \in \mathbb{R}$ . Find necessary and sufficient conditions for the existence of a convex body  $K \in \mathcal{K}_o^n$  such that  $\widetilde{C}_q(K, \cdot) = \mu$ .

An amazing feature of these dual curvature measures is that they also link two other well-known fundamental geometric measures of a convex body (cf. [25, Lemma 3.8]): when  $q = 0$  the dual curvature measure  $\widetilde{C}_0(K, \cdot)$  is –

up to a factor of  $n$  – Aleksandrov’s integral curvature of the polar body of  $K$  and for  $q = n$  the dual curvature measure coincides with the cone-volume measure of  $K$  given by

$$\tilde{C}_n(K, \eta) = V_K(\eta) = \frac{1}{n} \int_{\nu_K^{-1}(\eta)} \langle \mathbf{x}, \nu_K(\mathbf{x}) \rangle d\mathcal{H}^{n-1}(\mathbf{x}).$$

Similarly to the Minkowski problem, the dual Minkowski problem is equivalent to solving a Monge-Ampère type partial differential equation if the measure  $\mu$  has a density function  $g : S^{n-1} \rightarrow \mathbb{R}$ . In particular, if  $q \in (0, n]$ , then the dual Minkowski problem amounts to solving the Monge-Ampère equation

$$(1.7) \quad \frac{1}{n} h(\mathbf{x}) |\nabla h(\mathbf{x}) + h(\mathbf{x}) \mathbf{x}|^{q-n} \det[h_{ij}(\mathbf{x}) + \delta_{ij} h(\mathbf{x})] = g(\mathbf{x}),$$

where  $[h_{ij}(\mathbf{x})]$  is the Hessian matrix of the (unknown) support function  $h$  with respect to an orthonormal frame on  $S^{n-1}$ , and  $\delta_{ij}$  is the Kronecker delta.

If  $\frac{1}{n} h(\mathbf{x}) |\nabla h(\mathbf{x}) + h(\mathbf{x}) \mathbf{x}|^{q-n}$  were omitted in (1.7), then (1.7) would become the partial differential equation of the classical Minkowski problem, see, e.g., [9, 10, 38]. If only the factor  $|\nabla h(\mathbf{x}) + h(\mathbf{x}) \mathbf{x}|^{q-n}$  were omitted, then equation (1.7) would become the partial differential equation associated with the cone volume measure, the so-called logarithmic Minkowski problem (see, e.g., [7, 11]). Due to the gradient component in (1.7) if  $q \in (0, n)$ , the dual Minkowski problem is significantly more challenging than the classical Minkowski problem and logarithmic Minkowski problem.

The cone-volume measure for convex bodies has been studied extensively over the last few years in many different contexts, see, e.g., [2, 3, 6, 7, 8, 17, 19, 21, 25, 27, 28, 32, 33, 34, 35, 36, 37, 39, 43, 45, 46]. One very important property of the cone-volume measure – and which makes it so useful – is its  $\mathrm{SL}(n)$ -invariance, or simply called affine invariance. It is also the subject of the central *logarithmic Minkowski problem* which asks for sufficient and necessary conditions of a measure  $\mu$  on  $S^{n-1}$  to be the cone-volume measure of a convex body  $K \in \mathcal{K}_o^n$ . This is the  $p = 0$  limit case of the general  $L_p$ -Minkowski problem within the above mentioned  $L_p$  Brunn-Minkowski theory for which we refer to [26, 31, 47] and the references within.

The discrete, planar, even case of the logarithmic Minkowski problem, i.e., with respect to origin-symmetric convex polygons, was completely solved by Stancu [41, 42], and later Zhu [45] as well as Böröczky, Hegedűs and Zhu [4] settled (in particular) the case when  $K$  is a polytope whose outer normals are in general position.

In [7], Böröczky, Lutwak, Yang and Zhang gave a complete characterization of the cone-volume measure of origin-symmetric convex bodies among the even measures on the sphere. The key feature of such a measure is expressed via the following condition: A non-zero, finite Borel measure  $\mu$  on the unit sphere satisfies the *subspace concentration condition* if

$$(1.8) \quad \frac{\mu(S^{n-1} \cap L)}{\mu(S^{n-1})} \leq \frac{\dim L}{n}$$

for every proper subspace  $L$  of  $\mathbb{R}^n$ , and whenever we have equality in (1.8) for some  $L$ , there is a subspace  $L'$  complementary to  $L$ , such that  $\mu$  is concentrated on  $S^{n-1} \cap (L \cup L')$ .

Apart from the uniqueness aspect of the Minkowski problem, the symmetric case of the logarithmic Minkowski problem is settled.

**Theorem 1.1** ([7]). *A non-zero, finite, even Borel measure  $\mu$  on  $S^{n-1}$  is the cone-volume measure of  $K \in \mathcal{K}_e^n$  if and only if  $\mu$  satisfies the subspace concentration condition.*

An extension of the validity of inequality (1.8) to centered bodies, i.e., bodies whose center of mass is at the origin, was given in the discrete case by Henk and Linke [23], and in the general setting by Böröczky and Henk [5].

A generalization (up to the equality case) of the sufficiency part of Theorem 1.1 to the  $q$ -dual curvature measure for  $q \in (0, n]$  was given by Huang, Lutwak, Yang and Zhang. For clarity, we separate their main result into the next two theorems.

**Theorem 1.2** ([25, Theorem 6.6]). *If  $q \in (0, 1]$ , then an even finite Borel measure  $\mu$  on  $S^{n-1}$  is a  $q$ -dual curvature measure if and only if  $\mu$  is not concentrated on any great subsphere.*

**Theorem 1.3** ([25, Theorem 6.6]). *Let  $q \in [1, n]$  and let  $\mu$  be a non-zero, finite, even Borel measure on  $S^{n-1}$  satisfying the subspace mass inequality*

$$(1.9) \quad \frac{\mu(S^{n-1} \cap L)}{\mu(S^{n-1})} < 1 - \frac{q-1}{q} \frac{n - \dim L}{n-1}$$

*for every proper subspace  $L$  of  $\mathbb{R}^n$ . Then there exists a  $o$ -symmetric convex body  $K \in \mathcal{K}_e^n$  with  $\tilde{C}_q(K, \cdot) = \mu$ .*

In particular, it is highly desirable to understand how close (1.9) is to characterize  $q$ -dual curvature measures. Observe that for  $q = n$  the inequality (1.9) becomes essentially (1.8).

Our main result treats the necessity of a subspace concentration bound on dual curvature measures.

**Theorem 1.4.** *Let  $K \in \mathcal{K}_e^n$ ,  $q \in [1, n]$  and let  $L \subset \mathbb{R}^n$  be a proper subspace. Then we have*

$$(1.10) \quad \frac{\tilde{C}_q(K, S^{n-1} \cap L)}{\tilde{C}_q(K, S^{n-1})} \leq \min \left\{ \frac{\dim L}{q}, 1 \right\},$$

*and equality holds in (1.10) if and only if  $q = n$  and  $\tilde{C}_n(K, \cdot)$ , i.e., the cone-volume measure of  $K$ , satisfies the subspace concentration condition (1.8).*

In particular, for  $q < n$  we always have strict inequality in (1.10), but this is also optimal.

**Proposition 1.5.** *Let  $0 < q < n$  and  $k \in \{1, \dots, n-1\}$ . There exists a sequence of convex bodies  $K_l \in \mathcal{K}_e^n$ ,  $l \in \mathbb{N}$ , and a  $k$ -dimensional subspace  $L \subset \mathbb{R}^n$  such that*

$$\lim_{l \rightarrow \infty} \frac{\tilde{C}_q(K_l, S^{n-1} \cap L)}{\tilde{C}_q(K_l, S^{n-1})} = \begin{cases} \frac{k}{q} & , k \leq q, \\ 1 & , k \geq q. \end{cases}$$

We observe that if  $q \in [1, n]$  and  $\dim L = 1$  for a linear subspace  $L$  then  $1 - \frac{q-1}{q} \frac{n-\dim L}{n-1} = \frac{\dim L}{q}$ . Therefore Theorems 1.3 and 1.4 complete the characterization of the  $q$ -dual curvature measures if  $n = 2$ .

**Corollary 1.6.** *If  $q \in [1, 2)$ , then an even finite Borel measure  $\mu$  on  $S^1$  is a  $q$ -dual curvature measure if and only if*

$$\frac{\mu(S^1 \cap L)}{\mu(S^1)} < \frac{1}{q}$$

for every one-dimensional subspace  $L$  of  $\mathbb{R}^2$ .

We remark that the dual Minkowski problem is far easier to handle for the special case where the measure  $\mu$  has a positive continuous density, (where subspace concentration is trivially satisfied). The singular general case for measures is substantially more delicate, which involves measure concentration and requires far more powerful techniques to solve.

The paper is organized as follows. First we will briefly recall some basic facts about convex bodies needed in our investigations in Section 2. In Section 3 we will prove a lemma in the spirit of the celebrated Brunn-Minkowski theorem, which is one of the main ingredients for the proof of Theorem 1.4 given in Section 4. Finally, in Section 5 we will prove Proposition 1.5.

## 2. PRELIMINARIES

We recommend the books by Gardner [15], Gruber [20] and Schneider [40] as excellent references on convex geometry.

For a given convex body  $K \in \mathcal{K}^n$  the support function  $h_K: \mathbb{R}^n \rightarrow \mathbb{R}$  is defined by

$$h_K(\mathbf{x}) = \max_{\mathbf{y} \in K} \langle \mathbf{x}, \mathbf{y} \rangle.$$

A boundary point  $\mathbf{x} \in \partial K$  is said to have a (not necessarily unique) unit outer normal vector  $\mathbf{u} \in S^{n-1}$  if  $\langle \mathbf{x}, \mathbf{u} \rangle = h_K(\mathbf{u})$ . The corresponding supporting hyperplane  $\{\mathbf{x} \in \mathbb{R}^n: \langle \mathbf{x}, \mathbf{u} \rangle = h_K(\mathbf{u})\}$  will be denoted by  $H_K(\mathbf{u})$ . For  $K \in \mathcal{K}_o^n$  the radial function  $\rho_K: \mathbb{R}^n \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}$  is given by

$$\rho_K(\mathbf{x}) = \max\{\rho > 0: \rho \mathbf{x} \in K\}.$$

Note, that the support function and the radial function are homogeneous of degrees 1 and  $-1$ , respectively, i.e.,

$$h_K(\lambda \mathbf{x}) = \lambda h_K(\mathbf{x}) \text{ and } \rho_K(\lambda \mathbf{x}) = \lambda^{-1} \rho_K(\mathbf{x}),$$

for  $\lambda > 0$ . We define the reverse radial Gauss image of  $\eta \subseteq S^{n-1}$  with respect to a convex body  $K \in \mathcal{K}_o^n$  by

$$\alpha_K^*(\eta) = \{\mathbf{u} \in S^{n-1}: \rho_K(\mathbf{u})\mathbf{u} \in H_K(\mathbf{v}) \text{ for a } \mathbf{v} \in \eta\}.$$

If  $\eta$  is a Borel set, then  $\alpha_K^*(\eta)$  is  $\mathcal{H}^{n-1}$ -measurable (see [40, Lemma 2.2.11.]) and so the  $q$ -th dual curvature measure given in (1.6) is well defined. We will need the following identity.

**Lemma 2.1.** *Let  $K \in \mathcal{K}_o^n$ ,  $q > 0$  and  $\eta \subseteq S^{n-1}$  a Borel set. Then*

$$\tilde{C}_q(K, \eta) = \frac{q}{n} \int_{\mathbf{x} \in K, \mathbf{x}/|\mathbf{x}| \in \alpha_K^*(\eta)} |\mathbf{x}|^{q-n} d\mathcal{H}^n(\mathbf{x}).$$

*Proof.* Since  $q > 0$ , we may write via spherical coordinates

$$\begin{aligned} \frac{q}{n} \int_{\mathbf{x} \in K, \mathbf{x}/|\mathbf{x}| \in \alpha_K^*(\eta)} |\mathbf{x}|^{q-n} d\mathcal{H}^n(\mathbf{x}) &= \frac{1}{n} \int_{\alpha_K^*(\eta)} \left( \int_0^{\rho_K(\mathbf{u})} q r^{n-1} r^{q-n} dr \right) d\mathcal{H}^{n-1}(\mathbf{u}) \\ &= \frac{1}{n} \int_{\alpha_K^*(\eta)} \rho_K(\mathbf{u})^q d\mathcal{H}^{n-1}(\mathbf{u}) = \tilde{C}_q(K, \eta). \end{aligned}$$

□

Let  $L$  be a linear subspace of  $\mathbb{R}^n$ . We write  $K|L$  to denote the orthogonal projection of  $K$  onto  $L$  and  $L^\perp$  for the subspace orthogonal to  $L$ .

As usual, for two subsets  $A, B \subseteq \mathbb{R}^n$  and reals  $\alpha, \beta \geq 0$  the Minkowski combination is defined by

$$\alpha A + \beta B = \{\alpha \mathbf{a} + \beta \mathbf{b} : \mathbf{a} \in A, \mathbf{b} \in B\}.$$

By the well-known Brunn-Minkowski inequality we know that the  $n$ -th root of the volume of the Minkowski combination is a concave function. More precisely, for two convex bodies  $K_0, K_1 \subset \mathbb{R}^n$  and for  $\lambda \in [0, 1]$  we have

$$(2.1) \quad \text{vol}_n((1-\lambda)K_0 + \lambda K_1)^{1/n} \geq (1-\lambda)\text{vol}_n(K_0)^{1/n} + \lambda\text{vol}_n(K_1)^{1/n},$$

where  $\text{vol}_n(\cdot) = \mathcal{H}^n(\cdot)$  denotes the  $n$ -dimensional Hausdorff measure. We have equality in (2.1) for some  $0 < \lambda < 1$  if and only if  $K_0$  and  $K_1$  lie in parallel hyperplanes or they are homothetic, i.e., there exist a  $\mathbf{t} \in \mathbb{R}^n$  and  $\mu \geq 0$  such that  $K_1 = \mathbf{t} + \mu K_0$  (see, e.g., [14], [40, Sect. 6.1]).

### 3. INTEGRALS OF EVEN UNIMODAL FUNCTIONS

A function  $f$  on the real line  $\mathbb{R}$  is called unimodal if there is a number  $m \in \mathbb{R}$ , such that  $f$  is an increasing function on  $(-\infty, m)$  and decreasing on  $(m, \infty)$ . Obviously the integral of  $f$  over an interval of fixed length is maximal when the interval is centered at  $m$ . The notion of unimodal functions can be extended to higher dimensional spaces in the following way.

The superlevel sets of a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  are given by  $L_f^+(\alpha) = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \geq \alpha\}$ ,  $\alpha \in \mathbb{R}$ . We say that  $f$  is unimodal if every superlevel set of  $f$  is closed and convex. It was shown by Anderson [1] that the integral of an even unimodal function over translates of a symmetric convex region is maximal if the center of symmetry is moved to the origin. His proof relies only on the Brunn-Minkowski theorem. Here we generalize this approach to integrals over a convex combination of a convex body  $K$  and its reflection  $-K$ .

**Lemma 3.1.** *Let  $K \in \mathcal{K}^n$  and  $\dim K = k$ . Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$  be a unimodal function, such that  $f(\mathbf{x}) = f(-\mathbf{x})$  for every  $\mathbf{x} \in \mathbb{R}^n$  and*

$$\int_{\frac{1}{2}K + \frac{1}{2}(-K)} f(\mathbf{x}) d\mathcal{H}^k(\mathbf{x}) < \infty.$$

Let  $\lambda \in (0, 1)$ . Then

$$(3.1) \quad \int_{\lambda K + (1-\lambda)(-K)} f(\mathbf{x}) d\mathcal{H}^k(\mathbf{x}) \geq \int_K f(\mathbf{x}) d\mathcal{H}^k(\mathbf{x}).$$

Moreover, equality holds if and only if for every  $\alpha > 0$

$$\text{vol}_k\left([\lambda K + (1-\lambda)(-K)] \cap L_f^+(\alpha)\right) = \text{vol}_k(K \cap L_f^+(\alpha)).$$

*Proof.* Let  $K_\lambda = \lambda K + (1-\lambda)(-K)$ . By the convexity of  $L_f^+(\alpha)$  we have for every  $\alpha \in \mathbb{R}$

$$(3.2) \quad K_\lambda \cap L_f^+(\alpha) \supseteq \lambda(K \cap L_f^+(\alpha)) + (1-\lambda)((-K) \cap L_f^+(\alpha)).$$

The Brunn-Minkowski inequality (2.1) applied to the set on right hand side of (3.2) gives

$$\begin{aligned} \text{vol}_k(K_\lambda \cap L_f^+(\alpha)) &\geq \text{vol}_k\left(\lambda(K \cap L_f^+(\alpha)) + (1-\lambda)((-K) \cap L_f^+(\alpha))\right) \\ &\geq \left(\lambda \text{vol}_k(K \cap L_f^+(\alpha))^{1/k} + (1-\lambda) \text{vol}_k((-K) \cap L_f^+(\alpha))^{1/k}\right)^k. \end{aligned}$$

Since  $f$  is even, the superlevel sets  $L_f^+(\alpha)$  are symmetric. Hence,  $\text{vol}_k(K \cap L_f^+(\alpha)) = \text{vol}_k((-K) \cap L_f^+(\alpha))$  and so

$$\text{vol}_k(K_\lambda \cap L_f^+(\alpha)) \geq \text{vol}_k(K \cap L_f^+(\alpha))$$

for every  $\alpha \in \mathbb{R}$ . Fubini's theorem yields

$$\begin{aligned} \int_{K_\lambda} f(x) d\mathcal{H}^k(x) &= \int_0^\infty \text{vol}_k(K_\lambda \cap L_f^+(\alpha)) d\alpha \\ &\geq \int_0^\infty \text{vol}_k(K \cap L_f^+(\alpha)) d\alpha \\ &= \int_K f(x) d\mathcal{H}^k(x). \end{aligned}$$

Suppose we have equality in (3.1). Since  $\text{vol}_k(K_\lambda \cap L_f^+(\alpha))$  is continuous on the left with respect to  $\alpha$  we find that

$$\text{vol}_k(K_\lambda \cap L_f^+(\alpha)) = \text{vol}_k(K \cap L_f^+(\alpha))$$

for every  $\alpha > 0$ . □



## 4. PROOF OF THEOREM 1.4

Now we are ready to give the proof of Theorem 1.4. We use Fubini's theorem to decompose the dual curvature measure into integrals over hyperplane sections. Lemma 3.1 will provide a critical estimate for these integrals.

*Proof of Theorem 1.4.* In order to prove the inequality (1.10) we may certainly assume  $q > \dim L = k$ . For  $\mathbf{y} \in K|L$  put  $\overline{\mathbf{y}} = \rho_{K|L}(\mathbf{y})\mathbf{y}$  and

$$\begin{aligned} F_{\mathbf{y}} &= \text{conv}\{\mathbf{0}, K \cap (\overline{\mathbf{y}} + L^\perp)\}, \\ M_{\mathbf{y}} &= \text{conv}\{K \cap L^\perp, K \cap (\overline{\mathbf{y}} + L^\perp)\}. \end{aligned}$$

Observe that  $M_{\mathbf{y}} \cap (\overline{\mathbf{y}} + L^\perp) = F_{\mathbf{y}} \cap (\overline{\mathbf{y}} + L^\perp) = K \cap (\overline{\mathbf{y}} + L^\perp)$ . By Lemma 2.1 and Fubini's theorem we may write

$$\begin{aligned} (4.1) \quad \tilde{C}_q(K, S^{n-1}) &= \frac{q}{n} \int_{K|L} \left( \int_{K \cap (\overline{\mathbf{y}} + L^\perp)} |\mathbf{z}|^{q-n} d\mathcal{H}^{n-k}(\mathbf{z}) \right) d\mathcal{H}^k(\mathbf{y}) \\ &\geq \frac{q}{n} \int_{K|L} \left( \int_{M_{\mathbf{y}} \cap (\overline{\mathbf{y}} + L^\perp)} |\mathbf{z}|^{q-n} d\mathcal{H}^{n-k}(\mathbf{z}) \right) d\mathcal{H}^k(\mathbf{y}). \end{aligned}$$

In order to estimate the inner integral let  $\mathbf{y} \in K|L$ ,  $\mathbf{y} \neq \mathbf{0}$ , and for abbreviation we set  $\lambda = \rho_{K|L}(\mathbf{y})^{-1} \leq 1$ . Then by the symmetry of  $K$  we find

$$\begin{aligned} M_{\mathbf{y}} \cap (\overline{\mathbf{y}} + L^\perp) &\supseteq \lambda(K \cap (\overline{\mathbf{y}} + L^\perp)) + (1 - \lambda)(K \cap L^\perp) \\ &\supseteq \lambda(K \cap (\overline{\mathbf{y}} + L^\perp)) \\ &\quad + (1 - \lambda) \left( \frac{1}{2}(K \cap (\overline{\mathbf{y}} + L^\perp)) + \frac{1}{2}(-(K \cap (\overline{\mathbf{y}} + L^\perp))) \right) \\ &= \frac{1 + \lambda}{2}(K \cap (\overline{\mathbf{y}} + L^\perp)) + \frac{1 - \lambda}{2}(-(K \cap (\overline{\mathbf{y}} + L^\perp))). \end{aligned}$$

Hence the set  $M_{\mathbf{y}} \cap (\overline{\mathbf{y}} + L^\perp)$  contains a convex combination of a set and its reflection at the origin. This allows us to apply Lemma 3.1 from which we get

$$(4.2) \quad \int_{M_{\mathbf{y}} \cap (\overline{\mathbf{y}} + L^\perp)} |\mathbf{z}|^{q-n} d\mathcal{H}^{n-k}(\mathbf{z}) \geq \int_{K \cap (\overline{\mathbf{y}} + L^\perp)} |\mathbf{z}|^{q-n} d\mathcal{H}^{n-k}(\mathbf{z}).$$

Together with (4.1) we obtain the lower bound

$$(4.3) \quad \tilde{C}_q(K, S^{n-1}) \geq \frac{q}{n} \int_{K|L} \left( \int_{K \cap (\overline{\mathbf{y}} + L^\perp)} |\mathbf{z}|^{q-n} d\mathcal{H}^{n-k}(\mathbf{z}) \right) d\mathcal{H}^k(\mathbf{y}).$$

On the other hand we find

$$\begin{aligned}
\tilde{C}_q(K, S^{n-1} \cap L) &= \frac{q}{n} \int_{K|L} \left( \int_{F_{\mathbf{y}} \cap (\mathbf{y} + L^\perp)} |z|^{q-n} d\mathcal{H}^{n-k}(z) \right) d\mathcal{H}^k(\mathbf{y}) \\
&= \frac{q}{n} \int_{K|L} \left( \int_{(\rho_{K|L}(\mathbf{y}))^{-1}(K \cap (\overline{\mathbf{y}} + L^\perp))} |z|^{q-n} d\mathcal{H}^{n-k}(z) \right) d\mathcal{H}^k(\mathbf{y}) \\
&= \frac{q}{n} \int_{K|L} \rho_{K|L}(\mathbf{y})^{k-q} \left( \int_{K \cap (\overline{\mathbf{y}} + L^\perp)} |z|^{q-n} d\mathcal{H}^{n-k}(z) \right) d\mathcal{H}^k(\mathbf{y}).
\end{aligned}$$

The inner integral is independent of the length of  $\mathbf{y} \in K|L$  and might be as well considered as the value  $g(\mathbf{u})$  of a (measurable) function  $g: S^{n-1} \cap L \rightarrow \mathbb{R}_{\geq 0}$ . By taking this into account and using spherical coordinates we obtain

$$\begin{aligned}
\tilde{C}_q(K, S^{n-1} \cap L) &= \frac{q}{n} \int_{K|L} \rho_{K|L}(\mathbf{y})^{k-q} g(\mathbf{y}/|\mathbf{y}|) d\mathcal{H}^k(\mathbf{y}) \\
&= \frac{q}{n} \int_{S^{n-1} \cap L} g(\mathbf{u}) \left( \int_0^{\rho_{K|L}(\mathbf{u})} \rho_{K|L}(r\mathbf{u})^{k-q} r^{k-1} dr \right) d\mathcal{H}^{k-1}(\mathbf{u}) \\
&= \frac{q}{n} \int_{S^{n-1} \cap L} g(\mathbf{u}) \rho_{K|L}(\mathbf{u})^{k-q} \left( \int_0^{\rho_{K|L}(\mathbf{u})} r^{q-1} dr \right) d\mathcal{H}^{k-1}(\mathbf{u}) \\
&= \frac{1}{n} \int_{S^{n-1} \cap L} g(\mathbf{u}) \rho_{K|L}(\mathbf{u})^k d\mathcal{H}^{k-1}(\mathbf{u}).
\end{aligned}$$

Applying the same transformation to the right hand side of (4.3) gives

$$\begin{aligned}
\tilde{C}_q(K, S^{n-1}) &\geq \frac{q}{n} \int_{K|L} g(\mathbf{y}/|\mathbf{y}|) d\mathcal{H}^k(\mathbf{y}) \\
(4.5) \quad &= \frac{q}{n} \int_{S^{n-1} \cap L} g(\mathbf{u}) \left( \int_0^{\rho_{K|L}(\mathbf{u})} r^{k-1} dr \right) d\mathcal{H}^{k-1}(\mathbf{u}) \\
&= \frac{q}{n} \frac{1}{k} \int_{S^{n-1} \cap L} g(\mathbf{u}) \rho_{K|L}(\mathbf{u})^k d\mathcal{H}^{k-1}(\mathbf{u}).
\end{aligned}$$

Combining (4.4) and (4.5) yields (1.10) in the case  $k = \dim L \leq q$ , i.e.,

$$\frac{\tilde{C}_q(K, S^{n-1} \cap L)}{\tilde{C}_q(K, S^{n-1})} \leq \frac{k}{q}.$$

Now suppose that the dual curvature measure of  $K$  satisfies the inequality (1.10) with equality with respect to a proper subspace  $L$ . Then we certainly

have  $\dim L < q$ , since the curvature measure cannot be concentrated on a great hypersphere. Hence we must have equality in (4.2) for every  $\mathbf{y} \in \text{relint}(K|L)$ .

Assume  $q < n$ . Then the superlevel sets  $L_f^+(\alpha)$  of the function  $f(\mathbf{z}) = |\mathbf{z}|^{q-n}$ ,  $\mathbf{z} \in \mathbb{R}^n$ , are balls. Hence, in view of the equality condition of Lemma 3.1, equality in (4.2) implies that  $M_{\mathbf{y}} \cap (\mathbf{y} + L^\perp) \cap rB_n$  and  $K \cap (\bar{\mathbf{y}} + L^\perp) \cap rB_n$  have the same  $(n-k)$ -dimensional volume for every  $r > 0$ . For sufficiently small  $r$ , however, the intersection of  $K \cap (\bar{\mathbf{y}} + L^\perp) \subset \partial K$  with  $rB_n$  is empty. Hence we must have  $q = n$  and in this case we know by Theorem 1.1 that equality is attained if and only if the cone-volume measure of  $K$  satisfies the subspace concentration condition as stated in Theorem 1.1.  $\square$

**Remark 4.1.** *It is worth noting, that the proof of Theorem 1.4 only relies on the symmetry of the function  $|\cdot|^{q-n} = \rho_{B_n}(\cdot)^{n-q}$ , its homogeneity and the convexity of its unit ball. In fact, the ball  $B_n$  can be replaced by any symmetric convex body  $M \in \mathcal{K}_e^n$  in the sense that*

$$(4.6) \quad \int_{\alpha_K^*(S^{n-1} \cap L)} \rho_M(\mathbf{u})^{n-q} \rho_K(\mathbf{u})^q d\mathcal{H}^{n-1}(\mathbf{u}) \leq \frac{\dim L}{q} \int_{S^{n-1}} \rho_M(\mathbf{u})^{n-q} \rho_K(\mathbf{u})^q d\mathcal{H}^{n-1}(\mathbf{u}),$$

where  $L \subseteq \mathbb{R}^n$  is a subspace with  $\dim L \leq q$ . Observe, in this more general setting, Lemma 2.1 becomes

$$\begin{aligned} & \int_{\alpha_K^*(\eta)} \rho_M(\mathbf{u})^{n-q} \rho_K(\mathbf{u})^q d\mathcal{H}^{n-1}(\mathbf{u}) \\ &= \int_{\alpha_K^*(\eta)} \rho_M(\mathbf{u})^{n-q} \left( \int_0^{\rho_K(\mathbf{u})} q r^{n-1} r^{q-n} dr \right) d\mathcal{H}^{n-1}(\mathbf{u}) \\ &= q \int_{\mathbf{x} \in K, |\mathbf{x}| \in \alpha_K^*(\eta)} \rho_M(\mathbf{x})^{n-q} d\mathcal{H}^n(\mathbf{x}), \end{aligned}$$

and (4.6) can be proved along the same lines as Theorem 1.4 with  $\rho_{B_n}(\cdot)$  replaced by  $\rho_M(\cdot)$ .

## 5. PROOF OF PROPOSITION 1.5

Here we show that the bounds given in Theorem 1.4 are indeed tight for every choice of  $q \in (0, n)$ . To this end let  $k \in \mathbb{N}$  with  $0 < k < n$  and for  $r > 0$  let  $K_r$  be the cylinder

$$K_r = (rB_k) \times B_{n-k}.$$

Let  $L = \text{lin}\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$  be the  $k$ -dimensional subspace generated by the first  $k$  canonical unit vectors  $\mathbf{e}_i$ .

For  $\mathbf{x} \in \mathbb{R}^n$  write  $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ , where  $\mathbf{x}_1 \in \mathbb{R}^k \times \{\mathbf{0}\}$  and  $\mathbf{x}_2 \in \{\mathbf{0}\} \times \mathbb{R}^{n-k}$ . The supporting hyperplane of  $K_r$  with respect to a unit vector  $\mathbf{v} \in S^{n-1} \cap L$

is given by

$$H_{K_r}(\mathbf{v}) = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{v}_1, \mathbf{x}_1 \rangle = r\}.$$

Hence the part of the boundary of  $K_r$  covered by all these supporting hyperplanes is given by  $rS^{k-1} \times B_{n-k}$ . In view of Lemma 2.1 and Fubini's theorem we conclude

$$(5.1) \quad \tilde{C}_q(K_r, S^{n-1} \cap L) = \frac{q}{n} \int_{\mathbf{x}_1 \in rB_k} \left( \int_{\substack{\mathbf{x}_2 \in B_{n-k} \\ r|\mathbf{x}_2| \leq |\mathbf{x}_1|}} (|\mathbf{x}_1|^2 + |\mathbf{x}_2|^2)^{\frac{q-n}{2}} d\mathcal{H}^{n-k}(\mathbf{x}_2) \right) d\mathcal{H}^k(\mathbf{x}_1).$$

Denote the volume of  $B_n$  by  $\omega_n$ . Recall, that the surface area of  $B_n$  is given by  $n\omega_n$  and for abbreviation we set

$$c = c(q, k, n) = \frac{q}{n} k \omega_k (n - k) \omega_{n-k}.$$

Switching to the cylindrical coordinates

$$\mathbf{x}_1 = s\mathbf{u}, \quad s \geq 0, \mathbf{u} \in S^{k-1}, \quad \mathbf{x}_2 = t\mathbf{v}, \quad t \geq 0, \mathbf{v} \in S^{n-k-1},$$

transforms the right hand side of (5.1) to

$$(5.2) \quad \begin{aligned} \tilde{C}_q(K_r, S^{n-1} \cap L) &= c \int_0^r \int_0^{s/r} s^{k-1} t^{n-k-1} (s^2 + t^2)^{\frac{q-n}{2}} dt ds \\ &= c \int_0^r \int_0^1 s^{q-1} r^{k-n} t^{n-k-1} (1 + r^{-2} t^2)^{\frac{q-n}{2}} dt ds \\ &= c r^k \int_0^1 \int_0^1 s^{q-1} t^{n-k-1} (r^2 + t^2)^{\frac{q-n}{2}} dt ds. \end{aligned}$$

Analogously we obtain

$$(5.3) \quad \begin{aligned} \tilde{C}_q(K_r, S^{n-1}) &= \frac{q}{n} \int_{\mathbf{x}_1 \in rB_k} \left( \int_{\mathbf{x}_2 \in B_{n-k}} (|\mathbf{x}_1|^2 + |\mathbf{x}_2|^2)^{\frac{q-n}{2}} d\mathcal{H}^{n-k}(\mathbf{x}_2) \right) d\mathcal{H}^k(\mathbf{x}_1) \\ &= c \int_0^r \int_0^1 s^{k-1} t^{n-k-1} (s^2 + t^2)^{\frac{q-n}{2}} dt ds \\ &= c r^k \int_0^1 \int_0^1 s^{k-1} t^{n-k-1} (r^2 s^2 + t^2)^{\frac{q-n}{2}} dt ds. \end{aligned}$$

When  $q > k$ , the monotone convergence theorem gives

$$\lim_{r \rightarrow 0+} \int_0^1 \int_0^1 s^{q-1} t^{n-k-1} (r^2 + t^2)^{\frac{q-n}{2}} dt ds = \int_0^1 s^{q-1} ds \cdot \int_0^1 t^{q-k-1} dt = \frac{1}{q(q-k)}$$

and

$$\lim_{r \rightarrow 0+} \int_0^1 \int_0^1 s^{k-1} t^{n-k-1} (r^2 s^2 + t^2)^{\frac{q-n}{2}} dt ds = \int_0^1 s^{k-1} ds \cdot \int_0^1 t^{q-k-1} dt = \frac{1}{k(q-k)}.$$

Hence, by (5.2) and (5.3) we get

$$\lim_{r \rightarrow 0+} \frac{\tilde{C}_q(K_r, S^{n-1} \cap L)}{\tilde{C}_q(K_r, S^{n-1})} = \frac{k}{q}.$$

Now suppose  $q \leq k$ . Rewrite (5.3) as

$$(5.4) \quad \tilde{C}_q(K_r, S^{n-1}) = c r^k \int_0^1 \int_0^{1/s} s^{q-1} t^{n-k-1} (r^2 + t^2)^{\frac{q-n}{2}} dt ds,$$

which in view of (5.2) gives

$$(5.5) \quad \begin{aligned} & \tilde{C}_q(K_r, S^{n-1} \cap L) - \tilde{C}_q(K_r, S^{n-1}) \\ &= c r^k \int_0^1 \int_1^{1/s} s^{q-1} t^{n-k-1} (r^2 + t^2)^{\frac{q-n}{2}} dt ds. \end{aligned}$$

Observe, that by the monotone convergence theorem

$$(5.6) \quad \begin{aligned} \lim_{r \rightarrow 0+} \int_0^1 \int_1^{1/s} s^{q-1} t^{n-k-1} (r^2 + t^2)^{\frac{q-n}{2}} dt ds &= \int_0^1 \int_1^{1/s} s^{q-1} t^{q-k-1} dt ds \\ &= \begin{cases} \int_0^1 s^{q-1} \frac{1-s^{k-q}}{k-q} ds, & \text{if } q < k, \\ \int_0^1 s^{q-1} (-\log s) ds, & \text{if } q = k, \end{cases} \\ &= \frac{1}{kq}. \end{aligned}$$

On the other hand, if  $0 < r < 1$ , then

$$(5.7) \quad \begin{aligned} \int_0^1 \int_0^{1/s} s^{q-1} t^{n-k-1} (r^2 + t^2)^{\frac{q-n}{2}} dt ds &\geq \int_0^1 \int_r^1 s^{q-1} t^{n-k-1} (r+t)^{q-n} dt ds \\ &\geq \int_0^1 \int_r^1 s^{q-1} t^{n-k-1} (t+t)^{q-n} dt ds \\ &= \begin{cases} \frac{2^{q-n}}{q} \frac{r^{q-k}-1}{k-q}, & \text{if } q < k, \\ \frac{2^{q-n}}{q} (-\log(r)), & \text{if } q = k, \end{cases} \end{aligned}$$

which is not bounded from above as a function in  $r$ . Hence, by (5.4), (5.5), (5.6), (5.7) we finally get

$$\lim_{r \rightarrow 0+} \frac{\tilde{C}_q(K_r, S^{n-1} \cap L)}{\tilde{C}_q(K_r, S^{n-1})} = 1 - \lim_{r \rightarrow 0+} \frac{\int_0^1 \int_0^1 s^{q-1} t^{n-k-1} (r^2 + t^2)^{\frac{q-n}{2}} dt ds}{\int_0^1 \int_0^1 s^{q-1} t^{n-k-1} (r^2 + t^2)^{\frac{q-n}{2}} dt ds} = 1,$$

which finishes the proof of Proposition 1.5.

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